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Teaching fractions

by Lisa Fazio
and Robert Siegler

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Series Preface

The present booklet entitled "Teaching fractions" has been prepared for inclusion in the Educational Practices Series, a publication developed by the International Academy of Education (IAE). As part of its mission, the International Academy of Education provides timely syntheses of research on educational topics of international importance. The booklets are published and distributed by UNESCO's International Bureau of Education (IBE) and the Academy. This is the twenty-second in a series of booklets on educational practices that have been shown to improve learning.

The authors of the booklet are Lisa Fazio and Robert Siegler. Lisa Fazio is a post-doctoral researcher at Carnegie Mellon University, Pittsburgh, USA. Her research focuses on how adults and children learn new information and how they correct errors in their knowledge. She is currently investigating how children understand whole number and fraction magnitudes. Robert Siegler is Teresa Heinz Professor of Cognitive Psychology. He is well known for his research on children's thinking, problem-solving and reasoning. He has investigated for many years how children learn mathematics and how the theoretical understanding of mathematical development can be applied to improving mathematical learning, particularly in the case of pre-schoolers coming from low-income backgrounds. Professor Siegler is a member of the United States National Academy of Education, has served on the National Mathematics Advisory Panel (from 2006 to 2008), and headed the Fractions Practice Guide Panel for the United States Department of Education in 2009–2010.

The Academy is grateful to Professor Siegler and Dr Fazio for planning, drafting and revising the present booklet. The contents of the booklet are based on a report by the Institute of Educational Sciences of the United States Department of Education entitled *Developing effective fractions instruction: A practice guide* (Siegler et al., 2010). Understanding fractions is one of the most important skills that need to be developed in the mathematics curriculum. It is essential for understanding algebra and geometry and other aspects of mathematics. Yet, fractions have proven to be very difficult to understand for most students around the world. We hope that the present booklet will provide teachers with helpful information about how to facilitate their students' conceptual understanding of fractions.

The officers of the International Academy of Education are aware that this booklet is based on research carried out primarily in economically advanced countries. The booklet, however, focuses on students' difficulties with fractions that seem likely to be generally applicable throughout the world. Nevertheless, the recommendations of this booklet need to be assessed with reference to local conditions and adapted accordingly. In any educational setting, guidelines for practice require sensitive and sensible applications, and continuing evaluation of their effectiveness.

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Table of Contents

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Introduction

Students around the world have difficulties in learning about fractions. In many countries, the average student never gains a conceptual knowledge of fractions. For example, on a national test, only 50% of American 8th graders correctly ordered three fractions from smallest to largest (National Council of Teachers of Mathematics, 2007). Even in countries where the majority of students do achieve reasonably good conceptual understanding, such as Japan and China, fractions are considered a difficult topic. One reason for their difficulty is that fractions offer students their first lesson that many properties that are true of whole numbers are not true for all numbers. For example, with fractions, multiplication does not always lead to an answer larger than the multiplicands, division does not always lead to an answer smaller than the dividend, and numbers do not have unique successors. Overcoming the belief that properties that are true for whole numbers are true for all numbers is a major challenge, even in high-school many students do not understand that there are an infinite number of numbers between any two fractions (Vamvakoussi & Vosniadou, 2010). Yet, understanding fractions is essential for learning algebra, geometry and other aspects of higher mathematics.

This research guide provides suggestions for teachers and administrators looking to improve fraction instruction in their classrooms or schools. The recommendations are based on a synthesis of published research evidence produced by the United States Department of Education's Institute for Education Sciences, *Developing effective fractions instruction: A practice guide* (Siegler et al., 2010). The panel that produced the report included mathematics educators, mathematics teachers, mathematicians and psychologists. The recommendations are based on scientific research, along with the expertise and experience of successful mathematics educators.

The recommendations include a variety of classroom activities and teaching strategies, but all are focused on improving students' conceptual understanding of fractions. We define conceptual knowledge of fractions as knowledge of what fractions mean, for example their magnitudes and relations to physical quantities, an understanding of why arithmetic procedures with fractions are mathematically justified and why they yield the answers they do. Such conceptual knowledge can be contrasted with procedural knowledge—the ability to execute a series of steps to solve a problem. For example, a student might have the procedural knowledge to solve fraction division problems through inverting the divisor and multiplying the inverted divisor by the dividend, but might lack conceptual knowledge regarding why this procedure is mathematically justified and why it yields the answers it does.

Students' difficulties with fractions often stem from a lack of conceptual understanding. Many students view fractions as meaningless symbols or view the numerator and denominator as separate numbers rather than a unified whole. The recommendations presented here are designed to ensure that students understand fractions and can successfully solve computational problems involving them.

The guide starts with ideas for introducing fraction concepts in kindergarten and early elementary school, and continues with activities and teaching strategies designed to help older students understand fraction magnitudes and computational procedures involving fractions. It then examines ways of helping students use fractions to solve rate, ratio and proportion problems. The final recommendation suggests methods to increase teacher's conceptual knowledge of fractions. Teachers with a firm conceptual knowledge of fractions, along with knowledge of students' common errors and misconceptions, are essential for improving students' learning about fractions. Throughout the guide, we use the term "fraction" to encompass all of the ways of expressing rational numbers, including decimals, percentages and negative fractions.

Suggested readings: Hoffer et al., 2007; Moseley, Okamoto & Ishida, 2007; Mullis et al., 1997; Siegler, Thompson & Schneider, 2011.

1. Early introduction to fractions

Children should be introduced to fractions at an early age by building on their informal understanding of sharing and proportionality.

Research findings

Young children understand the concept of equal sharing. Four-year-olds can distribute a set of objects equally among a small number of recipients (e.g. twelve cookies shared among three people). By the age of 5, children can share a single object among multiple recipients (e.g. a candy bar). In addition, young children have an early understanding of proportional relations. For example, by the age of 6, children can match equivalent proportions represented by different geometric figures or everyday shapes (e.g. $^1/_2$ of a pizza is the same as $^1/_2$ of a box of chocolates). This early knowledge can be used to introduce the concept of fractions, connecting students' intuitive knowledge to formal fraction concepts.

The activities presented below can also be used to develop students' understanding of ordering and equivalence relations among fractions. In addition, the activities introduce students to two of the basic interpretations of fractions. Sharing activities can be introduced in terms of division—dividing eight candies into four equal groups—or they can be presented in terms of ratios—if three cookies are shared by two children, the ratio of cookies to children is 3:2. Fractions are typically introduced in 1st or 2nd grade, but the activities suggested below can be started as early as pre-school or kindergarten.

Equal-sharing activities

Teachers should begin with simple sharing activities that involve dividing a set of objects equally among a small group of people. The activities should involve sets of objects that can be shared equally among the recipients, with no remaining objects (e.g. six cookies shared by two people). The teacher can describe the number of objects and the number of recipients, and the student can determine how many objects each sharer would receive. Students should be encouraged to use concrete objects, drawings or other representations to help them solve the problems. Teachers should emphasize that each recipient needs to receive an equal number. As the children's knowledge improves, both the number of objects to be shared and the number of recipients can increase.

Next, students can move on to sharing problems that involve dividing objects into smaller parts. For example, if two people share one cookie, each person receives $^{1}/_{2}$ of a cookie. Instead of asking how many objects each person would receive, the question becomes how much of an object each person should have. Teachers can begin with sharing a single object among two or more recipients, and then move on to sharing multiple objects. An early question might involve four people sharing one apple, while a more advanced question might have four people sharing two apples. It is recommended that teachers start with sharing activities that allow children to use a halving strategy (dividing an object in half, dividing the new parts in half, etc.), before moving on to sharing activities that require children to divide objects into thirds or fifths. Teachers can also start introducing formal fraction names (e.g. one-half, one-third, one-quarter) and have the children label their drawings with the names.

Sharing activities can be used to help students understand the relative sizes of fractions. For example, by sharing an object among two, three, four or five people, students can see that as the number of sharers increases, the size of the piece that each person receives decreases. This idea can also be linked to fractions notation, such that the students learn that ¼ is less than $^{1}/_{3}$, which is less than $^{1}/_{2}$.

Proportional reasoning activities

Teachers can build on students' informal understanding of relative size to develop early proportional reasoning concepts. In the beginning, teachers should present problems that encourage students to think about qualitative proportional relations between pairs of objects. For example, the big bear goes in the big bed, and the little bear goes in the little bed. Another problem could be determining how many children it would take to balance a seesaw with one adult on one of the sides versus balancing a seesaw with two adults on one of the sides. Students can also mix stronger or weaker solutions of food colouring in order to compare different proportions of water and food colouring.

Suggested readings: Empson, 1999; Frydman & Bryant, 1988; Streefland, 1991.

2. Fractions are numbers

Students need to understand that fractions are numbers with magnitudes.

Research findings

Fractions are often taught using the idea that fractions represent a part of a whole. For example, one-fourth is one part of a whole that has been split into four parts. This interpretation is important, but it fails to convey the vital information that fractions are numbers with magnitudes. As such, fractions can be ordered from smallest to largest or be equivalent in value ($^1/_2 = {}^2/_4 = {}^3/_6$...). Children who only understand the part/whole approach to fractions often make errors, such as saying that $^4/_3$ is not a number because a person cannot be given four parts of an object that is divided into three parts. The common error of trying to add fractions by first adding the numerators and then adding the denominators stems in part from not understanding that fractions are numbers with magnitudes. Solely relying on a part/whole understanding of fractions often leaves children confused as to the meaning of fractions larger than 1 and to the meaning of negative fractions.

One effective way to ensure that students understand that fractions are numbers with magnitudes is to use number lines during instruction. Number lines can be applied to all fractions, and they illustrate that each fraction corresponds to a given magnitude.

Measurement activities

Teachers can use measurement activities to help students understand that fractions are numbers. By measuring objects, students can learn that fractions allow more precise measurement than do whole numbers alone. One hands-on activity would be to use fraction strips to measure different objects in the classroom. Students start with a strip of construction paper or card that represents a unit and use that strip to measure an object. When an object's length is not equal to a whole number of strips, the teacher can pass out strips that represent $^1/_2$, $^1/_4$, $^1/_3$ or other fractions of the whole. For example, to measure a pencil, a student might use two whole strips and one half strip. Students should realize that the size of the object does not change, but they can measure the object more accurately by using the shorter strips, or a combination of shorter strips and whole strips, rather than the whole strips alone. Teachers who use fraction strips should emphasize that the size of the object is defined by the size of the original strip (the unit). For example, if the class started with larger unit strips, then each object would be fewer fraction strips long than if it started with shorter unit strips.

Number lines

Teachers should have students locate and compare fractions on number lines. By placing different fractions on a number line, students can compare their magnitudes and see that some fractions, such as ¾ and $^6/_8$, are equivalent. Teachers can begin with number lines that have the fractions already marked, e.g. a 0-to-1 number line with the eighths marked. This additional step will eliminate difficulties that the students might have in properly segmenting the number line. Students should be asked to place both fractions whose locations are marked (e.g. $^6/_8$) and fractions whose denominator is a factor or a multiple of the unit fraction marked on the number line (e.g. ¾, $^{12}/_{16}$). It is also important to include fractions that are equivalent to whole numbers (e.g. $^8/_8$) so that students understand that whole numbers can be written as fractions.

To help students understand and compare fractions with different denominators, a number line can be marked with one unit above the line and another below. For example, if students are asked to compare $^3/_5$ and $^7/_8$, the teacher might divide the number line into fifths above the line and into eighths below the line. As students advance, these additional supports can be removed, and students can use number lines with minimal labels, e.g. just the endpoints labelled or the endpoints plus the middle of the line.

Number lines can also be used to expand children's concept of fractions to include negative fractions, and to include fractions with a value greater than 1, along with decimals and percents. For example, students can be asked to place fractions such as $^{14}/_3$ on a number line that runs from 0 to 5 with each whole number marked, or to place positive and negative fractions on a number line with –1 on the left, +1 on the right, and 0 in the middle. A similar strategy can be used to help children connect the concepts of fractions, decimals and percents. Students can place a variety of numbers such as $^3/_5$, .25 and 33% on the same 0 to 1 number line. In order to emphasize that fractions and decimals can represent the same magnitude, students could place fractions and their decimal equivalents on the same number line.

Finally, number lines can be useful for demonstrating the idea of fraction density. One way that fractions differ from whole numbers is that there are an infinite number of fractions between any two other fractions. This concept can be difficult for students to understand, but can be illustrated using number lines. For example, students can start with a number line representing a whole number (e.g. 0 to 1) and divide that line into halves. They can then continue to halve the halves creating fourths/quarters, then eighths, then sixteenths, and so on. In this way the students can learn that any fraction can be partitioned into even smaller fractions.

Suggested readings: Rittle-Johnson, Siegler & Alibali, 2001; Siegler, Thompson & Schneider, 2011; Stafylidou & Vosniadou, 2004.

3. Manipulatives and visual representations of fractions

Visual representations of fractions help develop conceptual understanding of computational procedures.

Research findings

Students are often taught computational procedures without adequate explanation of why the procedures work. Yet, research has shown a positive correlation between students' conceptual understanding of fractions and their success in using procedures to solve problems. Children who understand why a common denominator is necessary when adding fractions are more likely to remember the correct procedure than children who do not understand why common denominators are required. Thus, teachers should focus on developing conceptual understanding along with procedural fluency. One way to improve conceptual understanding is to use manipulatives and visual representations of fractions. Studies that have taught fraction arithmetic using visual representations of fractions have shown positive effects on students' computational skills.

Addition and subtraction

Visual representations can be used to help illustrate the need for common denominators when adding and subtracting fractions. For example, a teacher can demonstrate addition by using fractions of an object (e.g. $1/3$ of a rectangle and $1/2$ of a rectangle). By placing the $1/3$ of the rectangle and the $1/2$ rectangle together inside a third rectangle, the teacher can show the approximate sum. She can then show that $1/3$ of the rectangle equals $2/6$, and that $1/2$ equals $3/6$, and that the sum is exactly $5/6$ of the rectangle. This type of concrete demonstration can help students to understand why common denominators are necessary when adding and subtracting fractions.

Multiplication

Pictorial representations can help students understand how multiplying fractions involves finding a fraction of a fraction. For example, to illustrate $1/4$ times $2/3$, a student can start with a rectangle, divide it into thirds vertically, and then shade $2/3$ of the rectangle with vertical lines. She would then divide the rectangle into fourths/quarters with three horizontal lines, and shade $1/4$ of the already shaded area with horizontal lines. At the end, two of the twelve small rectangles would be shaded both horizontally and vertically, representing the answer $^2/_{12}$. This procedure can be used to illustrate that the product of two fractions that are both less than 1 is always smaller than either of the original fractions. It also can be used to demonstrate the need to redefine the unit when multiplying fractions. At first, the child treats the entire rectangle as the whole, shading in $^2/_3$ of the area. Then, the child treats the shaded area as the whole, examining the number of equal size rectangles within the $^2/_3$ unit that were shaded when the $^2/_3$ unit was divided into quarters. Finally, the child returns to treating the whole as the unit, and notes that two of the twelve units within the whole were shaded both ways.

Division

One way to conceptualize division is to think of how many times the divisor can go into the dividend. For example, $^1/_2 \div {}^1/_4$ is the same question as "How many ¼s are in $^1/_2$?" Fraction strips can be used to illustrate fraction division. For the above problem, students could have two strips of equal length, one divided into halves and the other in fourths/quarters. The students can then discover how many ¼s will fit in $^1/_2$. This activity can also be done using number lines. The teacher would draw a number line with the halves labelled above the line and fourths/quarters labelled below. Again, the students could see that there are $^2/_4$ in $^1/_2$.

Suggested readings: Hudson Hawkins, 2008; Nishida, 2008.

4. Estimation before computation

Students should be taught to estimate answers to problems before computing the answers, so that they can judge the reasonableness of their computed answers.

Research findings

Many students' errors with fraction arithmetic can be avoided if they estimate their answers before attempting to use a formal algorithm. Estimation of fractions, however, does not come easily to many students. By practising estimation, students improve both their knowledge of fraction magnitudes and their understanding of fraction arithmetic. Estimation forces students to think out their answers and focuses students on what it means to add or multiply fractions, rather than just following a memorized rule without understanding.

Classroom activities

When students are solving fraction arithmetic problems, they can be asked to estimate the answer and explain their reasoning before computing the answer. By checking if their computed answers are reasonable, students can recognize when they either used an incorrect computational procedure or made a mistake executing a correct procedure. For example, a student may estimate that $^1/_2 + ^1/_3$ must be larger than $^1/_2$ but smaller than 1, because $^1/_3$ is smaller than $^1/_2$ and $^1/_2 + ^1/_2 = 1$. If the student then incorrectly calculates that $^1/_2 + ^1/_3 = ^2/_5$, the teacher can point out that the answer cannot be correct, because $^2/_5$ is less than $^1/_2$. Then the teacher can encourage the student to determine whether the wrong answer resulted from executing the procedure incorrectly or using an incorrect procedure and, when the latter turns out to be the case, the student can be encouraged to generate or recall a correct procedure.

Strategies for estimating with fractions

One estimation strategy is to use benchmarks. Benchmarks should be fractions that students are comfortable with such as 0, $^1/_2$ and 1. Students can then decide whether any given fraction between 0 and 1 is closest to 0, $^1/_2$ or 1. For example, if asked to add $^6/_7$ and $^5/_8$, a student might reason that $^6/_7$ is close to 1 and $^5/_8$ is close to $^1/_2$, so the answer should be close to 1 $^1/_2$.

Students can also consider the size of relevant unit fractions (fractions with 1 as the numerator) when estimating. Once children understand that unit fractions decrease in size when the denominator increases (e.g. $^1/_6$ is smaller than $^1/_5$), they can use this knowledge to help them estimate. For example, if asked to estimate $^7/_8 + ^1/_9$, students can reason that $^7/_8$ is $^1/_8$ away from 1, and since $^1/_9$ is smaller than $^1/_8$ the answer will be slightly less than 1.

Suggested readings: Behr, Post & Wachsmuth, 1986; Cramer & Wyberg, 2009.

5. Directly confront common fraction arithmetic misconceptions

Teachers should discuss and correct common misconceptions about fraction arithmetic.

Research findings

Children often confuse the rules of whole number arithmetic with those of fraction arithmetic. It is important to identify misconceptions about fractions arithmetic and to directly address why the misconceptions lead to incorrect answers, as well as why the correct procedures lead to correct answers. Teachers can lead group discussions about different computational procedures and why some lead to correct answers while others do not. The teacher can then emphasize that, except for multiplication, the procedures that work with whole number arithmetic do not work with fraction arithmetic. Students will gain greater conceptual understanding of fraction arithmetic when they understand why the procedures from whole numbers do not work, rather than just learning a new procedure for fractions.

Common misconceptions

Many children's fraction arithmetic reflects one or more common misconceptions. We discuss three of the most common misconceptions here and what teachers can do to correct each of them.

Treating fractions' numerators and denominators as separate whole numbers. When asked to subtract fractions, students often subtract the numerators and then subtract the denominators (e.g. $^5/_8 - ^1/_4 = ^4/_4$). These students are failing to treat the fraction as a unified number and instead treat the numerator and denominator as separate whole numbers. Teachers can help students to overcome the misconception that this is an acceptable procedure by presenting meaningful problems in the classroom. For example, they could ask: "If you have ¾ of an orange and give $^1/_3$ of the original orange to a friend, what fraction of an orange do you have left?" Students operating under the misconception described earlier in this paragraph will answer "$^2/_1$" or "2". Asking such students whether it makes sense that they could begin with ¾ of an orange, give part of it away, and wind up with two oranges should make clear the problems with the misconception. After being shown why their procedure is faulty, students should be more receptive to learning the correct procedures.

Leaving the denominator unchanged in fraction multiplication problems. When multiplying fractions with equal denominators, students often leave the denominator unchanged (e.g. $^4/_5$ x $^1/_5$ = $^4/_5$). This error may occur because students are presented with more fraction addition problems than fraction multiplication problems, leading students to incorrectly generalize to multiplication the addition procedure for addends with equal denominators. Teachers can correct this error by reminding students that the problem can be reframed as "$^4/_5$ of $^1/_5$". Because the problem is asking for a part of $^1/_5$, the answer cannot be larger than $^1/_5$.

Misunderstanding mixed numbers. Students often have difficulties solving problems with mixed numbers. Some students ignore the fractional parts and instead only focus on the whole number (e.g. $4\ ^2/_3 - 1\ ^2/_5 = 3$). Others decide that the whole numbers in the problem must have the same denominator as the fractions (e.g. $3 - ^2/_5 = ^3/_5 - ^2/_5 = ^1/_5$). A related misconception is adding the whole number to the numerator of the fractional part (e.g. $2\ ^2/_5$ x $^5/_6$ = $^4/_5$ x $^5/_6$ = $^{20}/_{30}$). All of these errors reflect a fundamental misunderstanding of what mixed numbers are and of the magnitudes they represent. Teachers should be sure to use proper fractions, improper fractions and mixed numbers in the classroom, and to translate often between mixed numbers and improper fractions.

Suggested readings: Ashlock, 2010; Mack, 1995, Stafylidou & Vosniadou, 2004.

6. Real-world contexts

Teachers should present fraction problems in real-world contexts with plausible numbers.

Research findings

Children are better able to solve fraction arithmetic problems when the problems are presented in meaningful, real-world contexts. Providing a real-world context encourages children to use their intuitive problem-solving strategies, rather than relying on memorized procedures. There are many sources of real-world contexts for fractions, including food, drink and time, and measurement tools such as watches and rulers. Students themselves can be a useful source of possible real-world contexts, allowing teachers to further personalize problems.

Connections between real-world problems and fraction notation

Children may correctly answer a problem in a real-world context, and yet compute an incorrect answer when the problem is written in fraction notation. For example, a child may know that two halves make a whole, but say that $^1/_2 + ^1/_2 = ^2/_4$. Teachers should emphasize the connection between real-world problems and the fraction notation used to represent the problem. Students can practice creating real-world contexts for problems presented in fraction notation and practice translating real-world problems into fraction notation. This type of activity will help connect students' intuitive knowledge of fractions with the formal notation.

Suggested readings: Anand & Ross, 1987; Irwin, 2001.

7. Proportional reasoning

Students should understand proportional reasoning before being taught the cross-multiplication algorithm.

Research findings

In order to successfully navigate advanced topics in mathematics, students need to understand proportional reasoning. Proportional reasoning is particularly important for understanding rates, ratios and proportions—three interpretations of fractions that are not taught as often as the part/whole representation. Further, proportional reasoning is often needed in everyday life. Scaling a recipe up or down, calculating the supplies needed for a home improvement project, and determining the unit price for a store-bought item are real-world activities that require the use of proportional reasoning.

It is important that students learn to solve proportional-reasoning problems using their own intuitive strategies before they are taught the cross-multiplication algorithm. In fact, even after students learn the algorithm, teachers should continue to talk about the students' informal reasoning strategies and how they result in the same answer as the algorithm. The reason is that once students learn the cross-multiplication algorithm, they often ignore the meaning of the problems, which leads to incorrect uses of the algorithm. Teachers can emphasize that the cross-multiplication algorithm makes possible solutions to problems that would be difficult to solve using other strategies, but students should have a conceptual grounding as to what proportional reasoning is, rather than just knowing how to solve problems by cross-multiplication.

Teachers should be aware that students often fail to generalize their problem-solving strategies across problem types. Parallel problems with different cover stories will often be treated differently by students who do not recognize the underlying similarity between the problems. Noting the underlying similarities between superficially dissimilar problems is therefore advisable.

Build on intuitive notions of proportional reasoning

Before children are taught how to solve ratio, rate and proportion problems with the cross-multiplication algorithm, they should learn to solve some of these problems using more intuitive strategies. Ideally, students will discover these strategies on their own, and then the class can discuss each strategy's strengths and weaknesses. If students do not discover these strategies without help, teachers can present story problems that encourage their discovery. For example, the problem: "Paul can buy three cookies for $2; how many cookies can he get with $6?" This encourages the use of the build-up strategy. Children can repeatedly add three more cookies and $2, so they get six cookies for $4 and nine cookies for $6. Another common informal strategy is to use unit ratios. When presented with the question: "Julie bought five toy cars for $25; how much would four cars cost?", students can reason that one car costs $5, so four cars would cost $20.

Once students understand informal strategies for solving proportional reasoning problems, the cross-multiplication algorithm can be introduced as a method for dealing with larger numbers and/or problems that are not easily solved using the build-up and unit-ratio strategies. Students should continue to solve problems using all three methods, and the class can discuss when each strategy is most useful. Students should also solve some problems using multiple strategies, so that they can see that they yield the same answer.

Multiple contexts

Teachers should use multiple contexts when presenting rate, ratio and proportion problems. A key goal is getting students to understand the underlying similarities between similar problems that are presented in different contexts. Teachers should help students to identify the key features of the problems and how similar methods can be used to solve superficially different problems. As discussed in Recommendation 6, problems should be presented using real-life contexts that are meaningful to students. For example, students can learn to compare prices by looking at the unit price. If one can buy four candy bars for $3, or six candy bars for $4.25, which purchase is better value? Other contexts for proportional reasoning problems include enlarging or reducing the size of a photo, and altering a recipe so it will feed more people or fewer people.

Suggested readings: Ahl, Moore & Dixon, 1992; Cramer, Post & Currier, 1993.

8. Teacher understanding

Professional development programmes should focus on improving teachers' understanding of fractions and how to teach them.

Research findings

To teach fractions well, teachers themselves must have a thorough understanding of fraction concepts and operations. Researchers have found that students' math achievement is positively correlated with the mathematical knowledge of their teacher. Unfortunately, many teachers lack a deep conceptual understanding of fractions, especially fraction arithmetic. This deep understanding is particularly important when using visual representations to teach fraction concepts. Teachers must be able to use a number of different representations and be able to select an appropriate representation for each situation. Teachers should also be knowledgeable about the types of errors and misconceptions that students are likely to generate during fraction instruction. When teachers know why their students are having difficulties, they can address students' misunderstanding directly.

A deeper understanding of fractions concepts

Teachers need to be able to explain not only how to solve a problem, but also why the procedure is appropriate and why flawed approaches are inappropriate. This type of explanation requires a deep knowledge of fraction computation. Professional development opportunities should focus on cultivating this deeper level of knowledge in teachers. Teachers can be asked to explain why an algorithm works, or they can solve advanced problems that allow them to identify concepts they do not yet fully understand. For example, almost all teachers know that fraction division problems can be solved through the procedure "invert and multiply". However, many teachers lack a deep understanding of why that procedure is effective.

It is important for teachers to know not only the fraction concepts that are taught at their grade level, but also the concepts that come before and after. By understanding what students have been taught earlier, teachers can build upon what the students already know and better identify the sources of students' misconceptions. Understanding the material that will be taught later helps teachers to know what the students need to learn during the current year to provide a solid base for subsequent instruction.

A deep understanding of fraction concepts is also necessary to use visual representations effectively in the classrooms. During professional development, teachers should be taught how to use representations effectively and how the representations connect to the concepts being taught. Different representations are more or less appropriate for different situations. For example, diagrams can be useful in explaining sharing scenarios (e.g. splitting three cookies into five equal pieces) and in understanding fraction division. Number lines can help students to focus on the magnitudes of fractions, while area models (rectangles or circles that are partially shaded) can illustrate part/whole representation of fractions. Teachers should also be familiar with the difficulties that may arise when using visual representations. For example, students may have trouble drawing equal shares, or they may misinterpret the length of a number line, placing $^1/_2$ at the midpoint of a 0 to 5 number line, instead of placing it near 0.

The ability to access fraction knowledge

Through professional development activities, teachers should learn about how students develop an understanding of fraction concepts and the difficulties that they face in properly understanding fractions. This discussion should be framed around research that has been conducted on fractions learning, along with teachers' classroom observations. One way for teachers to gain a deeper understanding of how children learn to represent fractions is to examine students' written work and video tapes of them solving fraction problems. Teachers could discuss among themselves why students are having difficulties with specific types of problems at different points in the students' knowledge about the development of fractions. Teachers should also discuss the types of errors that students commonly make and what misconceptions underlie each of the errors. This type of discussion can also allow teachers to learn what types of fraction problems they should ask their students to solve in order to diagnose the sources of students' misunderstandings. Once teachers understand why their students are having problems, they can address the specific misunderstandings in the classroom.

Suggested readings: Hill, Rowan & Ball, 2005; Ma, 1999; Vamvakoussi & Vosniadou, 2010.

Conclusion

Fractions are an important stepping-stone for learning advanced mathematics; they are also commonly used in everyday life. Yet, many students continue to struggle with fractions, even after years of instruction. We believe that the recommendations presented in this booklet will help strengthen fractions teaching, and will increase the number of students who understand fractions and correctly solve fraction arithmetic problems. It is important to note that fractions are a difficult topic. Even after implementing the recommendations in this booklet, students will not instantly understand what fractions are and how they differ from whole numbers. However, with repeated applications of the ideas and varied examples, these recommendations will improve students' understanding of fractions. The recommendations are designed to be implemented as a whole, although each recommendation should produce gains in conceptual understanding even if implemented alone.

The underlying principle behind all of these recommendations is that students need a deep conceptual understanding of fractions in order to learn about them effectively and to remember what they have learned. When students have a superficial understanding of fractions, the fraction symbol itself is meaningless, and the procedures used in fraction arithmetic seem arbitrary and easy to confuse with each other. By cultivating conceptual understanding, teachers can help students to understand that fractions are real numbers and that fraction arithmetic is a meaningful procedure rather than a series of arbitrary steps. Conceptual understanding is difficult to acquire, but it is vital for ensuring a deep and enduring understanding of fractions and fraction arithmetic.

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Notes

Notes

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